7 Fourier Transforms

(Lecture with S. Helgason)

Fourier series are defined as

$$f(x) \sim \sum a_n e^{inx}, \qquad a_n = \frac{1}{2\pi} \int_\pi^\pi f(x) e^{-inx} dx$$

for 2π period functions. Now take we can take an arbitrary interval, then our dense expo-
nentials are $\frac{1}{\sqrt{2\pi}} e^{(in\pi x)/A}$.

Then for $L^2(-A, A)$

$$f(x) \sim \sum_{-\infty}^{\infty} \frac{1}{\sqrt{2A}} e^{\frac{in x \pi}{A}} \int_{-A}^{A} f(y) \frac{1}{\sqrt{2\pi}} e^{-\frac{in\pi x}{A}} dy$$

Let

$$/ \qquad \int_{-A}^{A} f(y) e^{\frac{-in\pi y}{A}} dy = g\left(\frac{n\pi}{A}\right)$$

Then $f(x) \sim \frac{1}{2\pi} \sum_{-\infty}^{\infty} g(n\pi/A) e^{(in\pi x)/A}(\pi/A)$, then think of $n\pi/A$ as a new variable $u = n\pi/A$, $du = \frac{\pi}{\Delta}$, and let $A \to \infty$ define

$$g(u) = \int_{\mathbb{R}} f(x) e^{-ixu} dx$$

and we hope that

$$f(x) \sim \frac{1}{2\pi} \int_{\mathbb{R}} g(u) e^{ixu} du$$

Definition. $f \in L^1(\mathbb{R})$ define the **fourier transform** $\hat{f}$ by

$$\boxed{\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-ixy}dx, \qquad y \in \mathbb{R}}$$

$\hat{f}$ is bounded and uniformly continuous because

$$\|\hat{f}(y)\| \leq \int_{\mathbb{R}} |f(x)||e^{-ixy}|dx \leq \int_{\mathbb{R}} |f(x)|dx < \infty \qquad f \in L^1(\mathbb{R})$$

and to show uniform continuity

$$|\hat{f}(y+h) - \hat{f}(y)| = \left| \int_{\mathbb{R}} f(x)(e^{ix(y+h)} - e^{ixy})dx \right| = \left| \int_{\mathbb{R}} f(x)e^{ixy}(e^{ixh} - 1)dx \right|$$

$$\leq \int_{\mathbb{R}} |f(x)||e^{ixh} - 1|dx \leq 2|f(x)|$$

so by Lebesgue dominated convergence theorem, we can take limits

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x)||e^{ixh} - 1|dx = \int_{\mathbb{R}} \lim_{h \to 0} |f(x)||e^{ixh} - 1|dx = 0$$

so f is continuous, and in fact uniformly continuous, because of the bound $2|f(x)|$ (the y dropped out).

Definition. Schwarz Space A function f is a Schwarz function if for each $m \geq 0$, $l \geq 0$

$$x^m \frac{d^l f}{dx^l}, \qquad \text{bounded on } \mathbb{R}$$

Note 3 facts $C_c^\infty(\mathbb{R}) \subset S$, $e^{-x^2/2} \in S$ and $S \subset L^1(\mathbb{R})$. Also, it is not too hard to see by the definition that if $f \in S(\mathbb{R})$ then

$$|f(x)| \leq C(1 + |x|^2)^{-N}$$

for some C.

Lemma. $f \in S$, $g(x) = xf(x)$, *then*

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-ixy}dx, \qquad \frac{d}{dy}\hat{f}(y) = \int_{\mathbb{R}} f(x)(-ix)e^{-ixy}dx$$

In fact $\int f(x)(-ix)e^{-ixy}dx$ is uniformly convergent, and now

$$\left(\frac{d\hat{f}}{dy} \right)(y) = -i\widehat{xf}(y), \qquad \widehat{xf}(y) = i\frac{d\hat{f}}{dy}$$

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Lemma. *$f \in S$, $h = df/dx$ Then by integration by parts*

$$\hat{h}(y) = \widehat{\frac{df}{dx}} = \int_{\mathbb{R}} \left(\frac{df}{dx} \right) e^{-ixy} dx = \left[f(x) e^{-ixy} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)(-iy)(e^{-ixy}) dx$$

and as $x \to \infty$, $f(x) \to 0$, so then the expression above winds up being $iy\hat{f}(y)$. Thus

$$\left(\widehat{\frac{df}{dx}} \right)(y) = iy\hat{f}(y).$$

So sum up what we have done above in the following theorem

Theorem.

$$\widehat{t^k f(t)}(\xi) = (-i)^k \frac{d^k}{d\xi^k} \hat{f}(\xi) \qquad \widehat{\frac{d^n f(t)}{dt^n}}(\xi) = (i\xi)^n \hat{f}(\xi)$$

This helps in differential equations, the fourier transforms interchanges differentiation and multiplication, and vice versa. But the function we deal with has to be small at infinity, however the Schwarz space allows for more general fourier transforms, like polynomials.

However we don't know that functions are fourier transforms of L^1, that is $\widehat{L^1(\mathbb{R})} = ?$. However, for Schwarz functions...

Theorem. *$f \to \hat{f}$ is a bijection of S onto S and*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy$$

*This is the **fourier inversion formula**.*

Before the proof we present an example

Example. *f the characteristic function of $[-1, 1]$. Then*

$$\hat{f}(y) = \int_{-1}^{1} e^{-iyx} dx = 2\frac{\sin y}{y}$$

Then we would like to show that

$$f = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\sin y}{y} e^{iyx} dy$$

But there is a snag at $y = 0$. Consider

$$\int_{-A}^{A} \frac{\sin y}{y} (\cos yx + i \sin yx) dy = \int_{-A}^{A} \frac{\sin y}{y} \cos yx \ dy$$

(since $i \sin yx \cdot (\sin y/y)$ is odd). So then to take care of $y = 0$

$$2 \int_0^A \frac{\sin y}{y} \cos y = 2 \int_0^A \frac{\frac{1}{2}\sin(x+1)y}{y} + \frac{\frac{1}{2}\sin(x-1)y}{y}dy$$

But

$$\int_0^\infty \frac{\sin \alpha y}{y} = sgn\ \alpha$$

so as $A \to \infty$ we get $sgn(x+1) - sgn(x-1) = \chi_{[-1,1]}$.

We continue with the proof of the theorem

Proof. If $f \in S$, then $\hat{f} \in S$. And the following are both fourier transforms of Schwarz functions:

$$\boxed{\left(\frac{d}{dy}\hat{f}\right) = \pm i\widehat{yf(y)}, \qquad \left(\widehat{\frac{df(y)}{dy}}\right) = iy\hat{f}(y)}$$

By iterations of these we can show that $y^m \frac{d^l \hat{f}}{dy^l}$ is bounded, in fact

$$y^m \frac{d^n \hat{f}}{dy^n}$$

is the fourier transform of a Schwarz function, so it is bounded and thus $\hat{f}$ is a Schwarz function. $\qquad \square$

Assume the inversion formula is proved. Then $f \mapsto \hat{f}$ is one-to-one, also $f \mapsto \hat{f}$ is surjective because $g \in S$, $\check{g} = f$, $\hat{f} = g$, since $\hat{f} \mapsto f$ is one-to-one. Now we need to prove the inversion formula.

First we find the fourier transform of $e^{-\frac{x^2}{2}} = f(x)$. This is the solution to the differential equation

$$xf + \frac{df}{dx} = 0$$

Now if h is a solution then

$$\frac{d}{dx}(e^{\frac{x^2}{2}}h) = e^{\frac{x^2}{2}}\left(xh + \frac{dh}{dx}\right) = 0$$

So $e^{-\frac{x^2}{2}}h$ is constant, then $h = Ce^{-\frac{x^2}{2}}$, now

$$\left(\widehat{xf + \frac{df}{dx}}\right) = i\frac{d\hat{f}}{dy} + iy\hat{f}(y) = 0$$

So $\hat{f}$ satisfies the same differential equation as f, so $\hat{f}(y) = Ce^{-\frac{y^2}{2}}$. Calculate this at the origin to find the constant $\hat{f}(0) = \sqrt{2\pi}$, so

$$\hat{f}(y) = \sqrt{2\pi}e^{-\frac{y^2}{2}}$$

We need a few lemmas to finish the proof

Lemma. *Symmetry Lemma* $g, f \in S$ *then*

$$\int_{\mathbb{R}} f(x)\hat{g}(x)dx = \int_{\mathbb{R}} \hat{f}(y)g(y)dy$$

Proof. We can just use Fubini's theorem with

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)e^{-ixy}dx \right) g(y)dy$$

$\square$

Lemma. $f \in S$, $a \in \mathbb{R}$, $f^a(x) = f(x+a)$ *then*

$$\hat{f}(y+a) = \hat{f^a}(y) = e^{iay}\hat{f}(y)$$

Proof. Just by definition of the transform and change of variables.

Lemma. $f \in S$, $f_a(x) = f(x/a)$, *then*

$$\hat{f}(y/a) = \hat{f_a}(x) = a\hat{f}(ay)$$

Proof. Same as above

$\square$

Now a special case of the symmetry law gives

$$\int_{\mathbb{R}} \hat{f}(y)e^{-\frac{y^2}{2a^2}}dy = \sqrt{2\pi}a\int_{\mathbb{R}} f(x)e^{-\frac{a^2x^2}{2}}dx$$

Now if we do a change of variables and let $s = ax$, then the above becomes

$$\int_{\mathbb{R}} \hat{f}(y)e^{-\frac{y^2}{2a^2}}dy = \int_{\mathbb{R}} f\left(\frac{s}{a}\right)e^{-\frac{s^2}{2}}ds$$

Let $a \to \infty$. Then the above becomes

$$\int_{\mathbb{R}} \hat{f}(y)dy = \sqrt{2\pi}f(0)\int_{\mathbb{R}} e^{-\frac{s^2}{2}}ds = 2\pi f(0)$$

so the inversion holds at point 0. Use lemma about $f^a(x)$, so

$$f^a(0) = f(a) = \int_{\mathbb{R}} e^{iay}\hat{f}(y)dy$$

$\square$

Theorem. *Let $f \in S$, then*

$$2\pi \|f(x)\|_{L^2} = 2\pi \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(y)|^2 dy = \|\hat{f}(y)\|_{L^2}$$

Proof. Observe

$$\overline{f(x)} = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{\hat{f}(y)}e^{-ixy}dy = \frac{1}{2\pi}(\widehat{\overline{\hat{f}}})(x).$$

take $g = \overline{(\hat{f})}$. Then by symmetry $\hat{g} = \overline{f}(x)$, and

$$2\pi \int_{\mathbb{R}} f(x)\overline{f}(x)dx = \int_{\mathbb{R}} \hat{f}(y)\overline{\hat{f}}(y)dy$$

$\square$

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